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Research Article

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Analytic approximate solution for the Bratu's problem by optimal homotopy analysis method

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Abstract

In This paper, we present analytic approximate solutions for Bratu's problem with high accuracy for different values of λ. We solve this nonlinear problem without any approximations or transformation in the problem and we successfully obtain the two branches of solutions for different values λ using homotopy analysis method. A new efficient approach is proposed to obtain the optimal value of convergence controller parameter $\hbar$ to guarantee the convergence of the obtained series solution.

Keywords: Optimal homotopy analysis method , Bratu's problem, Series solutions, Two boundary value problem, Multiple solutions.

1 Introduction

Consider the Bratu's problem in one-dimensional planar:

$$\frac{d^2u}{dx^2} + \lambda e^u = 0,\ u(0) = u(1) = 0,\ \lambda > 0 \tag{1}$$

The Bratu's problem (1) nonlinear two boundary value problem with strong nonlinear term e^u and parameter λ, this problem appears in a number of applications such as the fuel ignition model of the thermal combustion theory, the model of thermal reaction process, the Chandrasekhar model of the expansion of the Universe, questions in geometry and relativity about the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [1, 2, 3].
The analytical solution of (1) in the following form:

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$$u(x) = -2\ln\left(\frac{cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{cosh\left(\frac{\theta}{4}\right)}\right) \tag{2}$$

Where θ is a solution of $\theta = \sqrt{2\lambda}\, cosh\left(\frac{\theta}{4}\right)$. The Bratu's problem has zero, one or two solutions when $\lambda > \lambda_c$, $\lambda = \lambda_c$ and $\lambda < \lambda_c$ respectively, where the critical value λ_c satisfies the equation $1 = \frac{1}{4}\sqrt{2\lambda_c}\,\sinh\left(\frac{\theta}{4}\right)$ and also the critical value is given by $\lambda_c = 3.513830719$ see [3]. Differentiating (2) with respect to x one time and setting $x = 0$ give,

$$u'(0) = \theta\tanh\left(\frac{\theta}{4}\right) \tag{3}$$

In this work, we solve The Bratu's problem (1) using the homotopy analysis method (HAM), This method initially proposed by Liao in his Ph.D, thesis [4] was proposed to get analytic approximations of highly nonlinear equations. The HAM can guarantee the convergence of the series solutions by auxiliary parameters especially the so-called convergence-controller parameter $\hbar$ [5-6]. In recent years, this method has been successfully employed to solve many types of nonlinear problems in science and engineering such as Troesch's problem [7], the Fitzhugh–Nagumo equation [8], heat radiation equations [9], MHD viscoelastic fluid flow [10], the Coupled nonlinear schrödinger equations [11], the Continuous population models for single and interacting species [12], the Boussinesq problem[13, 14], the Sturm–Liouville problems [15], the wave propagation problems [16], Laplace equation with Dirichlet and Neumann boundary conditions [17], differential-difference equation [18], fractional equations[19]. Many authors trying to solve equation (1) by homotopy analysis method [7, 20] but for λ=3 only. The basic motivation of this paper is to obtain analytical approximate solution of the Bratu's problem (1) by using homotopy analysis method for different values of λ and determined the optimal value of convergence-controller parameter $\hbar$ using the averaged residual error [21] .

2 Analysis of method

Consider the nonlinear two boundary value problems in finite domain:

$$u''(x) + f(u, u') = g(x) \tag{4}$$

with the boundary conditions

$$\left.\frac{d^{s_1}u(x)}{dx^{s_1}}\right|_{x=a} = A \quad , \quad \left.\frac{d^{s_2}u(x)}{dx^{s_2}}\right|_{x=b} = B\ ,\ s_{1,2} = 0\ or\ 1 \tag{5}$$

$f(u, u')$ is the nonlinear function, and $g(x)$ is the non-homogeneous term, we write equation (4) in the form,

$$N[u(x)] - g(x) = 0 \tag{6}$$

Where N is a nonlinear operator, x denote independent variable and $u(x)$ is an unknown function.The first step in this method is to add the new condition $\left.\frac{d^{1-s_1}u(x)}{dx^{1-s_1}}\right|_{x=a} = \epsilon$ or $\left.\frac{d^{1-s_2}u(x)}{dx^{1-s_2}}\right|_{x=b} = \epsilon$ where ϵ is unknown and will determine later, then the boundary conditions (5) become,

$$\left.\frac{d^{s_1}u(x)}{dx^{s_1}}\right|_{x=a} = A \quad , \quad \left.\frac{d^{1-s_1}u(x)}{dx^{1-s_1}}\right|_{x=a} = \epsilon \tag{7}$$

or

$$\left.\frac{d^{s_2}u(x)}{dx^{s_2}}\right|_{x=b} = B \quad , \left.\frac{d^{1-s_2}u(x)}{dx^{1-s_2}}\right|_{x=b} = \epsilon \tag{8}$$

the other boundary conditions $\left.\frac{d^{s_2}u(x)}{dx^{s_2}}\right|_{x=b} = B$ in (7) or $\left.\frac{d^{s_1}u(x)}{dx^{s_1}}\right|_{x=a} = A$ in (8) use to obtain as function of convergence-controller parameter $\hbar$. We construct the general zero-order deformation equation as follows:

$$(1-p)L[\phi(x,\epsilon,p) - u_0(x,\epsilon)] = p\hbar H(x)\big(N[\phi(x,\epsilon,p)] - g(x)\big) \tag{9}$$

Where $p \in [0,1]$ denote the so-called embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator. The HAM is based on a kind of continuous mapping $u(x,\epsilon) \to \phi(x,\epsilon,p)$; $\phi(x,\epsilon,p)$ is an unknown function, $u_0(x,\epsilon)$ is an initial guess of $u(x,\epsilon)$ and $H(x)$ denotes a non-zero auxiliary function. It is obvious that when the embedding parameter $p = 0$ and $p = 1$, Equation (6) becomes,

$$\phi(x,\epsilon,0) = u_0(x,\epsilon) \quad , \ \phi(x,\epsilon,1) = u(x,\epsilon) \tag{10}$$

Respectively. Thus as p increases from 0 to 1, the solution $\phi(x,\epsilon,p)$ varies from the initial guess $u_0(x,\epsilon)$ to the solution $u(x,\epsilon)$. Expanding $\phi(x,\epsilon,p)$ in the Taylor series with respect to p, one has,

$$\phi(x,\epsilon,p) = u_0(x,\epsilon) + \sum_{m=1}^{+\infty} u_m(x,\epsilon)p^m \tag{11}$$

Where

$$u_m(x,\epsilon) = \frac{1}{m!}\left.\frac{\partial^m \phi(x,\epsilon,p)}{\partial p^m}\right|_{p=0} \tag{12}$$

The initial guess $u_0(x,\epsilon)$ of the solution $u(x,\epsilon)$ can be determined by the rule of solution expression as follows. From the problem (4), with the new boundary condition (7) or (8), the solution $u(x,\epsilon)$ expressed by a set of base functions,

$$\{(x-c)^n | n = 0,1,2,3 \ldots.\} \quad , c = a \ or \ b \tag{13}$$

in the form

$$u(x,\epsilon) = \sum_{n=0}^{+\infty} f_n(\epsilon)(x-c)^n \tag{14}$$

The initial guess $u_0(x,\epsilon)$ can be chosen from equation (14) so that it achieves the boundary condition (7) or (8). The second goal is to determine the higher order terms $u_m(x,\epsilon)$ $(m, 1,2, \ldots.)$. Define the vector

$$\vec{u}_i(x) = \{u_0(x), u_1(x), \ldots, u_i(x)\} \tag{15}$$

Differentiating Equation (9) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$ we have the so-called mth-order deformation equation:

$$L[u_m(x,\epsilon) - \chi_m u_{m-1}(x,\epsilon)] = \hbar H(x) R(\vec{u}_{m-1}) \tag{16}$$

and its boundary condition (17) or (18) for the new boundary conditions (7) or (8) respectively

$$\left.\frac{d^{s_1}u_m(x)}{dx^{s_1}}\right|_{x=a} = 0 \quad , \left.\frac{d^{1-s_1}u_m(x)}{dx^{1-s_1}}\right|_{x=a} = 0 \tag{17}$$

$$\left.\frac{d^{s_2}u_m(x)}{dx^{s_2}}\right|_b = 0 \quad , \quad \left.\frac{d^{1-s_2}u_m(x)}{dx^{1-s_2}}\right|_b = 0 \qquad , \; m \geq 1 \tag{18}$$

where

$$R(\vec{u}_{m-1}) = \frac{1}{m-1!}\left.\frac{\partial^{m-1}(N[\phi(x,\epsilon,p)]-g(x))}{\partial p^{m-1}}\right|_{p=0} \tag{19}$$

and

$$\chi_m = \begin{cases} 0 & when \;\; m \leq 1 \\ 1 & otherwise \end{cases} \tag{20}$$

Now the solution of the mth-order deformation equation (16) for $m \geq 1$ when $H(x) = 1$ becomes

$$u_m(x,\epsilon) = \chi_m u_{m-1}(x,\epsilon) + L^{-1}\big(\hbar R(\vec{u}_{m-1})\big) \tag{21}$$

then

$$u(x,\epsilon) \;\cong U_M(x,\epsilon,\hbar) = \sum_{m=0}^{M} u_m(x,\epsilon,\hbar) \tag{22}$$

The third goal is to determine the optimal value of convergence-controller parameter $\hbar$, from equation (22) and unused boundary conditions from (5) in new boundary condition (7) or (8), it,

$\left.\frac{d^{s_2}u(x)}{dx^{s_2}}\right|_{x=b} = B$ or $\left.\frac{d^{s_1}u(x)}{dx^{s_1}}\right|_{x=a} = A$, we can be find the relate between the convergence-control

parameter $\hbar$ and ϵ .

$$\left.\frac{d^{s_2}U_M(x,\epsilon,\hbar)}{dx^{s_2}}\right|_{x=b} = B \tag{23}$$

or

$$\left.\frac{d^{s_1}U_M(x,\epsilon,\hbar)}{dx^{s_1}}\right|_{x=a} = A \tag{24}$$

By plotting the equation (23) or (24) given the set $R_\hbar$ for the convergence-control parameter that where the value of constant ϵ. And using any $\hbar \,\epsilon\, R_\hbar$ one can get a convergent series solution. However, the convergence rate is also dependent upon $\hbar$ but the so-called $\hbar$ -curve approach cannot give the "optimal" value of $\hbar$ in $R_\hbar$. We can use the so-called averaged residual error [21] defined by

$$E_M = \frac{1}{k}\sum_{s=0}^{k}(N(U_M(s\Delta x,\epsilon,\hbar) - g(s\Delta x))^2 \tag{25}$$

The minimum averaged residual error is given by a nonlinear algebraic equation

$$\frac{\partial E_M}{\partial \hbar} = 0 \tag{26}$$

and the residual error

$$R(x) = N[U_M(x,\epsilon,\hbar)] - g(x). \tag{27}$$

The new approach to get the optimal value of convergence-controller parameter $\hbar$ by solving the two equations (23) or (24) and (26). We will use this new approach to solve the nonlinear Bratu's problem (1) at different values of λ.

3 APPLICATION

Consider the nonlinear Bratu's problem

$$\frac{d^2u}{dx^2} + \lambda e^u = 0 \tag{28}$$

with boundary condition

$$u(0) = u(1) = 0 \tag{29}$$

H. N. Hassan et all; [7] and Abbasbandy et all; [20], applied successfully the HAM to obtain multiple branches of solutions of this nonlinear problem for $\lambda = 3$ only. In [7] the authors used the Taylor expansion of the strongly nonlinear term $e^u \cong 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \frac{u^4}{4!}$, and in [20] the authors use transformation $y(x) = e^{-u(x)}$, but in this paper, we solve this nonlinear problem without any approximations or transformation and solving the problem for different values λ. Multiplying equation (28) in $\frac{du}{dx}$ and integration to get

$$2\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 + \epsilon^2 + 2\lambda = 0 \tag{30}$$

where $\epsilon = u'(0)$. The new boundary conditions according (7) are

$$u(0) = 0\,, \quad u'(0) = \epsilon\ . \tag{31}$$

We choose the auxiliary linear operator

$$L[\phi(x,\epsilon,p)] = \frac{\partial^2\phi(x,\epsilon,p)}{\partial x^2} \tag{32}$$

with the property

$$L[c_0 + c_1 x] = 0 \tag{33}$$

we define the nonlinear operator as

$$N[\phi(x,\epsilon,p)] = 2\frac{\partial^2\phi(x,\epsilon,p)}{\partial x^2} - \left(\frac{\partial\phi(x,\epsilon,p)}{\partial x}\right)^2 + \epsilon^2 + 2\lambda \tag{34}$$

and

$$R_m(\vec{u}_{m-1}) = 2\frac{d^2u_{m-1}}{dx} - \sum_{j=0}^{m-1} u_j u_{m-1-j} + (1-\chi_m)(\epsilon^2 + 2\lambda) \tag{35}$$

The solution of the mth-order deformation equation (21) for $m \geq 1$ is

$$u_m(x,\epsilon) = \chi_m u_{m-1}(x,\epsilon) + \hbar \int\int R_m(\vec{u}_{m-1})\, dxdx + c_0 + c_1 x \tag{36}$$

Where the integration constants c_0 and c_1 are determined by the boundary conditions according (17) are,

$$u_m(0) = 0\,,\ \ u'_m(0) = 0 \tag{37}$$

We choose only initial guess $u_0(x,\epsilon)$ for all different values of λ, imposed according to initial condition (31) and equation (14)

$$u_0(x,\epsilon,\lambda) = \epsilon x - \frac{\lambda}{2}x^2 - \left(\frac{\epsilon\lambda}{6}\right)x^3 + \left(\frac{\lambda^2}{24} - \frac{\lambda}{24}\epsilon^2\right)x^4 \tag{38}$$

The equations (36) and (37) can be easily solved by symbolic computation software's such as Mathematica. For example the soluation of (36) at $m = 1$ is

$$u_1(x,\hbar,\epsilon,\lambda) = \frac{x^5\lambda(168\epsilon^3 - 672\epsilon\lambda)\hbar}{10080} + \frac{x^6\lambda(-196\epsilon^2\lambda + 112\lambda^2)\hbar}{10080} + \frac{x^7\lambda(-40\epsilon^3\lambda + 40\epsilon\lambda^2)\hbar}{10080} + \frac{x^8\lambda(-5\epsilon^4\lambda + 10\epsilon^2\lambda^2 - 5\lambda^3)\hbar}{10080}$$

$u_m(x,\hbar,\epsilon)$ $(m = 2,3,4,\ldots)$ Can be calculated similarly. The approximation solution $U_M(x,\hbar,\epsilon,\lambda)$ to the Bratu's problem (28) is,

$$U_M(x,\hbar,\epsilon,\lambda) = \sum_{\mathrm{m}=0}^{\mathrm{M}} \mathrm{u_m}(x,\hbar,\epsilon,\lambda) \tag{39}$$

To find the relation between the convergence-control parameter $\hbar$ and ϵ, using the boundary condition $u(1) = 0$ in equation (39), it become

$$u(1) \cong U_M(1,\hbar,\epsilon,\lambda) = 0 \tag{40}$$

We got ϵ as a function of $\hbar$ from (40) that is plotted in Fig.1, at $\lambda = 3$. Form Fig.1, two values of ϵ is clear: lower interval solution $\boldsymbol{R}_\hbar$ = [-0.6, -0.2] and upper interval solution $\boldsymbol{R}_\hbar$ = [-0.3, -0.2], but cannot give the optimal value of $\hbar$ of Fig.1, we can find this value by solving the equations (40) and (26), the point which lies inside the interval solution $\boldsymbol{R}_\hbar$ be the optimal value of $\hbar$ and thus becomes the minimum averaged residual error E_M (25) at this optimal value. Table.1, Shows that the optimal value of $\hbar$ for different values of k at $\lambda = 3$, we find that the lower optimal solution at $\hbar = -0.3808$ and the upper optimal solution at $\hbar = -0.2349$ at $k = 20$. Form this table clear the accurate value of ϵ at $k = 20$, by increasing the value of k, we get the more accurate solution, but this required the more CPU time.

$\lambda = 3$				
k	*Lower solution*		*Upper solution*	
	$\hbar$	ϵ	$\hbar$	ϵ
2	-0.3801	2.31960278	-0.2346	6.10243
10	-0.3083	2.31960277	-0.2347	6.10244
20	-.03808	2.31960276	-0.2349	6.10245
Exact solution (3)		2.31960225	Exact solution (3)	6.10338

Table 1: the optimal value of $\hbar$ for different value of k at $\lambda = 3$

One can see from Fig.2 and fig.3, the accuracy of the two different solutions obtained by the present method. The absolute residual $|R(x)|$(27) has maximum magnitude 0.00003 in lower solution case and 0.12 in the upper solution case, which show the accuracy of the approximate solution and the efficiency of the proposed method.

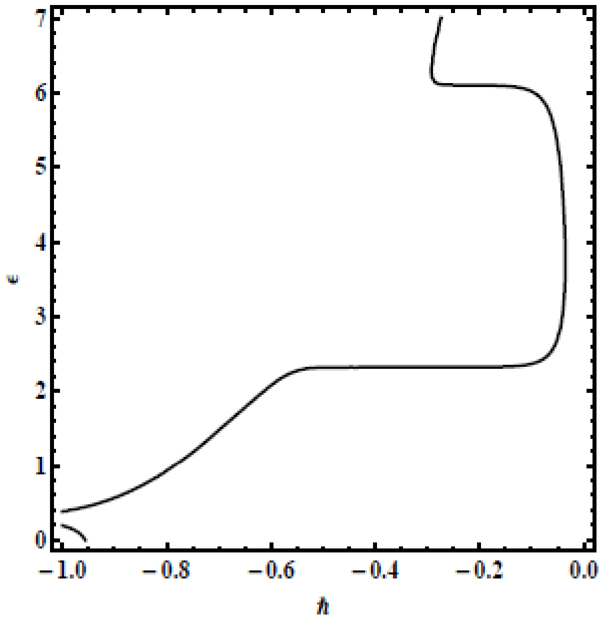

Figure 1: Plotting ϵ as a function of $\hbar$ at $\lambda = 3$ and $M = 15$ in (40)

The Bratu's problem (28) has been resolved in previous papers [7] and [20] at $\lambda = 3$ only, but in this paper successfully solved the problem with high accuracy for different values of λ. as shown in Fig.4, fig.5 and fig.6 show the problem solutions at $\lambda = \frac{5}{2}, 2$ and $\frac{3}{2}$ respectively, the sets lower and upper solutions $R_{\hbar}$ can be summarized by the table.2, the optimal values of $\hbar$ and the values of $u'(0) = \epsilon$ summarized in the table.3, table.4 and table.5, In all cases, we not only obtain the two values of the solution. but also, with high accuracy comparison with exact values.

λ	*Lower interval solution*	*Upper interval solution*
5/2	[-0.2,-0.6]	[-0.2,-0.3]
2	[-0.2,-0.7]	[-0.1,-0.3]
3/2	[-0.2,-0.8]	[-0.2,-0.4]

Table 2: The sets lower and upper $\boldsymbol{R}_{\hbar}$ for different values of λ

Fig.7, 8, 9, 10, 11, and fig.12, show the absolute residual $|R(x)|$ for different values of λ lower and upper solutions, table.6, shows the accuracy of the solution obtained by the present method, The absolute residual $|R(x)|$ has maximum magnitude in lower solutions between 4×10^{-12} and 1.2×10^{-7} and in upper solution between 0.07 and 0.8.

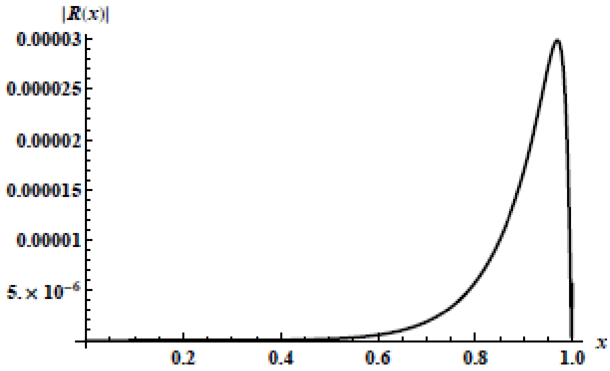

Figure 2: The absolute residual in the lower solution at $k = 20$ and $\lambda = 3$.

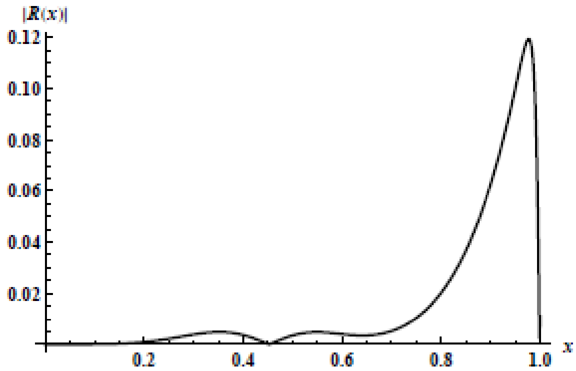

Figure 3: The absolute residual in upper solution at $k = 20$ and $\lambda = 3$

λ = 5/2				
k	*Lower solution*		*Upper solution*	
	$\hbar$	ϵ	$\hbar$	ϵ
2	- 0.42026	1.704359460	- 0.2361	7.2084
10	- 0.42087	1.704359460	- 0.2363	7.2084
20	- 0.42088	1.704359460	- 0.2367	7.2084
Exact solution (3)		1.704359459	Exact solution (3)	7.2093

Table 3: The optimal value of $\hbar$ for different value of k at $\lambda = 5/2$

λ = 2				
k	*Lower solution*		*Upper solution*	
	$\hbar$	ϵ	$\hbar$	ϵ
2	- 0.4472	1.24821751776	- 0.2626	8.26849
10	- 0.4476	1.24821751776	- 0.2635	8.26852
20	- 0.4480	1.24821751776	- 0.2654	8.2685
Exact solution (3)		1.248217517758	Exact solution (3)	8.2687

Table 4: The optimal value of $\hbar$ for different value of k at $\lambda = 2$

$\lambda = 3/2$				
k	*Lower solution*		*Upper solution*	
	$\hbar$	ϵ	$\hbar$	ϵ
2	- 0.4662	0.8732347855610143	- 0.2994	9.4234
10	- 0.4664	0.8732347855610142	- 0.3581	9.4236
20	- 0.4664	0.8732347855610135	- 0.3558	9.4236
Exact solution (3)		0.8732347855610132	Exact solution (3)	9.4238

Table 5: The optimal value of $\hbar$ for different value of k at $\lambda = 3/2$

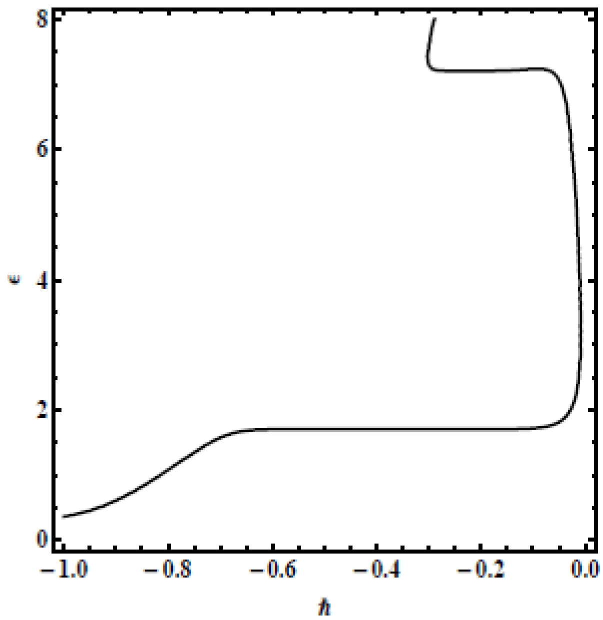

Figure 4: Plotting ϵ as a function of $\hbar$ at $\lambda = 5/2$ and $M = 15$ in (40)

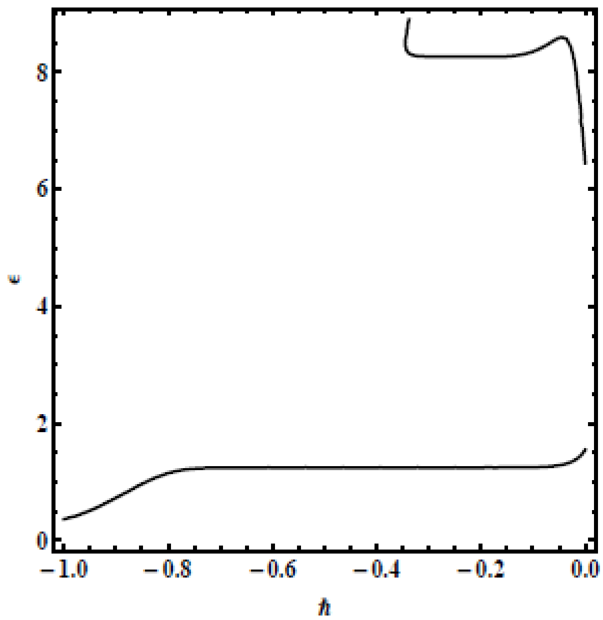

Figure 5: Plotting ϵ as a function of $\hbar$ at $\lambda = 2$ and $M = 15$ in (40)

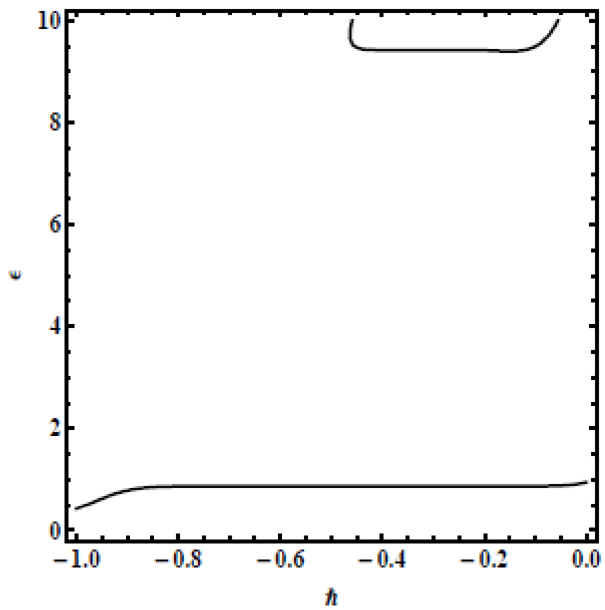

Figure 6: Plotting ϵ as a function of $\hbar$ at $\lambda = 3/2$ and $M = 15$ in (40)

λ	5/2	2	3/2
Max $\lvert R(x)\rvert$ of lower solution	$1.2\text{x}10^{-7}$	$2\ \text{x}10^{-10}$	$4\ \text{x}10^{-12}$
Max $\lvert R(x)\rvert$ of upper solution	0.07	0.17	0.8

Table 6: Maximum magnitude of $\lvert R(x)\rvert$ in both solutions for different values of λ

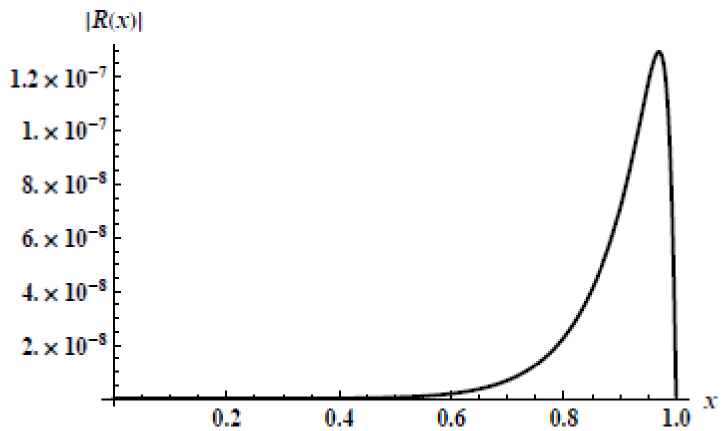

Figure 7: The absolute residual in lower solution at $k = 20$ and $\lambda = 5/2$

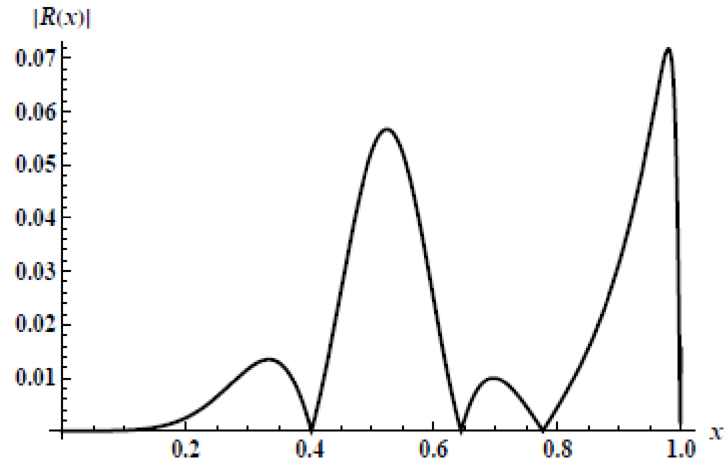

Figure 8: The absolute residual in upper solution at $k = 20$ and $\lambda = 5/2$

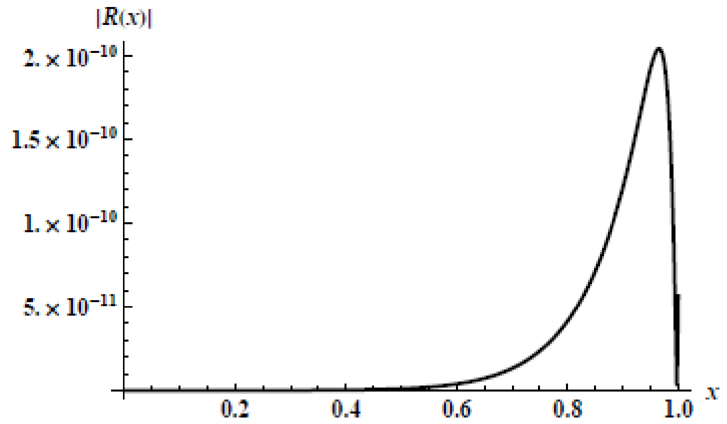

Figure 9: The absolute residual in lower solution at $k = 20$ and $\lambda = 2$

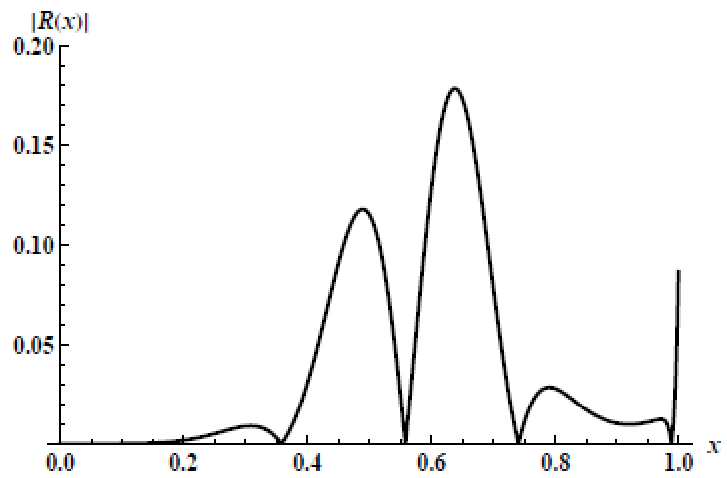

Figure 10: The absolute residual in upper solution at $k = 20$ and $\lambda = 2$

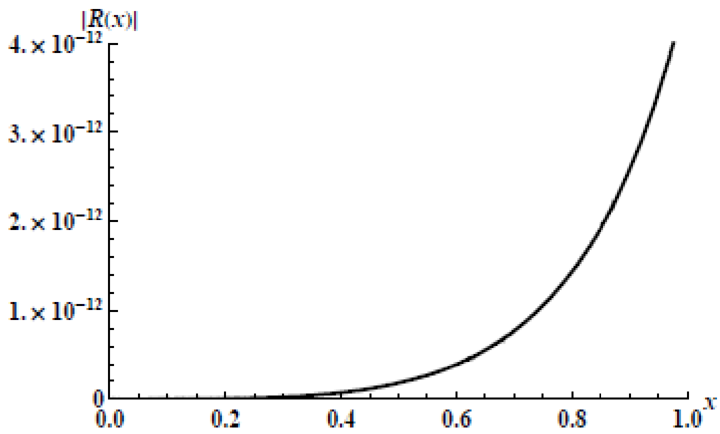

Figure 11: The absolute residual in Lower solution at $k = 20$ and $\lambda = 3/2$

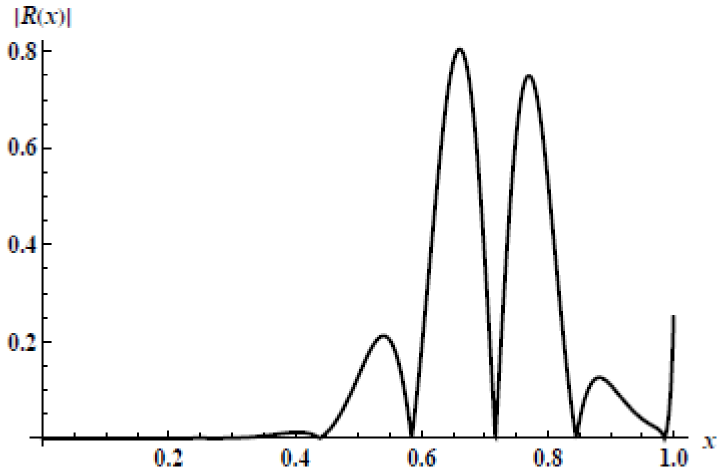

Figure 12: The absolute residual in upper solution at $k = 20$ and $\lambda = 3/2$

Conclusion

This paper successfully solved nonlinear two-boundary value problem appears in a number of applications. This problem cantains the strong nonlinear term e^u and parameter λ. The proposed method successfully in putting the new approach to get the optimal value of convergence-controller parameter $\hbar$, puts approach to solve the two-boundary ordinary differential equation with Neumann conditions, solving the Bratu's problem for different value of λ as acase study with high accuracy and solved the problem without replacing the nonlinear term by Taylor series or using any transformation

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