Representable good EQ-algebras

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Abstract

Recently, a special algebra called *EQ-algebra* (we call it here *commutative EQ-algebra* since its multiplication is assumed to be commutative) has been introduced by V. Novák in [22], which aims at becoming the algebra of truth values for *fuzzy type theory*. Its implication and multiplication are no more closely tied by the adjunction and so, this algebra generalizes commutative residuated lattice. One of the outcomes is the possibility to relax the commutativity of the multiplication. This has been elaborated by El-Zekey et al. in [8]. We continue in this paper the study of EQ-algebras (i.e., those with multiplication not necessarily commutative). We introduce *prelinear* EQ-algebras, in which the join-semilattice structure is not assumed. We show that every prelinear and good EQ-algebra is a lattice EQ-algebra. Moreover, the $\{\wedge, \vee, \rightarrow, 1\}$-reduct of a prelinear and separated lattice EQ-algebra inherits several lattice-related properties from product of linearly ordered EQ-algebras. We show that prelinearity alone does not characterize the *representable* class of all good (commutative) EQ-algebras. One of the main results of this paper is to characterize the representable good EQ-algebras. This is mainly based on the fact that $\{\rightarrow, 1\}$-reducts of good EQ-algebras are BCK-algebras and runs on lines of C. J. van Alten's [2] characterization of representable integral residuated lattices. We also supply a number of potentially useful results, leading to this characterization.

Key words: EQ-algebra, commutative residuated lattice, fuzzy equality, fuzzy logic, BCK-algebra, prelinearity, representable algebras

1. Introduction

Recently [22], a special algebra called *EQ-algebra* has been introduced which aims at becoming the algebra of truth values for fuzzy type theory (FTT) [21]. It has three basic operations – *meet* $\wedge$, *multiplication* $\otimes$ and *fuzzy equality* $\sim$ – and a *top element* 1, while the *implication* $\rightarrow$ is derived from fuzzy equality $\sim$.

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This research was supported by the grant obtained from SAIA under the National Scholarship Program (NSP) of the Slovak Republic.

In [23], the study of EQ-algebras have been further deepened. Moreover, the axioms originally introduced in [22] have been slightly modified.

It is interesting that the implication and multiplication, in EQ-algebra, are no more closely tied by the adjunction and so, this algebra generalizes residuated lattice and hence the multiplication has weaker properties. One of the outcomes is the possibility to relax the commutativity of the multiplication without loss of anything essential. This has been elaborated by El-Zekey et al. (see [8]) and opens an exciting possibility to develop a fuzzy logic with a non-commutative conjunction but a single implication only (see [24]). Accordingly, throughout this paper, by EQ-algebras we mean those with *non-commutative* multiplications. While EQ-algebras with commutative multiplications, i.e. as in [23], are called here *commutative EQ-algebras*.

It is also interesting, though, that adding the adjunction condition to EQ-algebra leads to a commutative residuated lattice and not to a non-commutative one. This means that EQ-algebras are not generalization of pseudo-residuated lattices. From the point of view of potential applications, it seems very interesting that unlike [17], we can have non-commutativity without necessity to introduce two kinds of implication. Thus, the applications especially in modeling of commonsense reasoning in natural language might be more natural.

We continue in this paper the study of EQ-algebras, begun in [8, 22, 23]. First, we review the basic definitions and properties of EQ-algebras and their special kinds.

We introduce prelinear EQ-algebras, in analogy with the prelinear structures of Esteva-Godo [9], Hájek [16] and Höhle [18]. It should be emphasized here that the underlying poset of an EQ-algebra need not be a join-semilattice. Nevertheless, the prelinearity condition merely states that 1 is the unique upper bound in E of the set $\{(a \to b), (b \to a)\}$. We show that every prelinear and good EQ-algebra is a good ℓEQ-algebra; i.e. lattice-ordered EQ-algebra with substitution property holding also for the operation $\vee$ (if it is defined). Moreover, we show that $\{\wedge, \vee, \to, 1\}$-reducts of prelinear and separated lattice EQ-algebra inherits several lattice-related properties from product of linearly ordered EQ-algebras.

One of the main results of this paper is to characterize the class of all good EQ-algebras that may be represented as subalgebras of products of linearly ordered good EQ-algebras. Such algebras are called *representable*. Our characterization is based on the fact that $\{\to, 1\}$-reducts of good EQ-algebras are BCK-algebras and runs on lines of C. J. van Alten's [2] characterization of representable integral residuated lattices. We show that prelinearity alone does not characterize representable good (commutative) EQ-algebras.

It should be emphasized here that axiomatizations of the class of representable algebras in some related classes of algebras have been considered in the literature, notably the representable commutative residuated lattices are characterized by the prelinearity. This result is contained in [11], although an earlier result by M. Pałasinski in [26] characterizes the representable BCK-algebras, which are the $\{\to, 1\}$-subreducts of commutative residuated lattices (see also [29]). C. J. van Alten [2] characterized representable integral residuated lattices (he used the name "biresiduated lattice"). An axiomatization of all representable residuated lattices (not only integral ones) which, however, uses all the operations of residuated lattices was independently obtained by K. Blount and C. Tsinakis [3].

We introduce and study in depth the prefilters, filters and the congruences of separated EQ-algebras and show that many properties of lattices of (pre)filters of residuated lattices can be obtained for lattices of (pre)filters of separated EQ-algebras, with the main result that the lattice of filters (which form a complete sublattice of the lattice of all prefilters) is isomorphic to the lattice of relative congruences. We also show that, in the case of good EQ-algebras, the lattice of filters are in bijective correspondence with the lattice of congruences and these lattices are distributive.

We prove that representable good EQ-algebras can be characterized by

$$(d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u \leq ((a \to b) \to u) \to u. \tag{1}$$

Moreover, we show that if the multiplication $\otimes$ is commutative then the inequality (1) is equivalent to

$$(c \to ((b \to a) \otimes c)) \to u \leq ((a \to b) \to u) \to u. \tag{2}$$

Consequently, the representable good and commutative EQ-algebras can be characterized by (2). The proof consists in showing that if a good (commutative) EQ-algebras satisfies (1) (satisfies (2), respectively), then every minimal prime prefilter is a filter. The proof is based on a detailed study of the prefilter lattice (see Section 4 below).

This paper is organized as follows. In Section 2, we review the basic definitions and properties of EQ-algebras and their special kinds. In Section 3, We introduce prelinear EQ-algebras. We introduce and study in depth the prefilters, filters and the congruences of EQ-algebras in Section 4. We devote Section 5 to characterize the representable class of good EQ-algebras. The results are summarized in Section 6.

2. EQ-algebras: An overview

2.1. Definition and examples

Definition 1 ([8])
An EQ-algebra is an algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$$

of type $(2, 2, 2, 0)$, *where for all* $a, b, c, d \in E$:

(E1) $\langle E, \wedge, \mathbf{1} \rangle$ is a $\wedge$-semilattice with top element $\mathbf{1}$. We put $a \leq b$ iff $a \wedge b = a$, as usual.

(E2) $\langle E, \otimes, \mathbf{1} \rangle$ is a monoid and $\otimes$ is isotone in both arguments w.r.t. $a \leq b$.

(E3) $a \sim a = \mathbf{1}$ (reflexivity)

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \leq (c \sim (d \wedge b))$ (substitution)

(E5) $(a \sim b) \otimes (c \sim d) \leq (a \sim c) \sim (b \sim d)$ (congruence)

(E6) $(a \wedge b \wedge c) \sim a \leq (a \wedge b) \sim a$ (isotonicity axiom)

(E7) $(a \wedge b) \sim a \leq (a \wedge b \wedge c) \sim (a \wedge c)$ (antitonicity axiom)

(E8) $a \otimes b \leq a \sim b$ (boundedness)

The operation "$\wedge$" is called meet (infimum), "$\otimes$" is called multiplication and "$\sim$" is a fuzzy equality.

Note that Definition 1 differs from the original definition of EQ-algebras (see [23, Definition 1]) in that the multiplication $\otimes$ needs not be commutative. Throughout this paper, EQ-algebras with commutative multiplications, i.e. as in [23], will be called *commutative EQ-algebras*. Although the original definition in [8] was stated without assuming both the associativity and the commutativity of the multiplication (the resulting algebras, in [8], has been called *semicopula-based EQ-algebras*), we are only interested in this particular case of semicopula-based EQ-algebras.

We will also put, for $a, b \in E$

$$a \to b = (a \wedge b) \sim a. \tag{3}$$

The derived operation (3) will be called *implication*. Using it, axioms (E6) and (E7), in fact, express isotonicity of $\to$ w.r.t. the second variable and antitonicity of $\to$ w.r.t. the first variable. We point out that the idea for developing mathematics on the basis of identity (equality) roots to G. W. Leibnitz and, e.g. the definition $a \to b = (a \wedge b) \sim a$, in fact, has been introduced by him (cf. [7]). Note also that the substitution axiom (E4) can be seen also as a special form of the extensionality (see, e.g. [16]).

The class of (commutative) EQ-algebras is a variety (see [8]). Consequently, the class of EQ-algebras is closed under direct products (with coordinatewise operations and order), subalgebras and homomorphic images; by virtue of a theorem on equational classes due to Birkhoff, see [15].

Definition 2 ([23])
Let $\mathcal{E}$ be an EQ-algebra. We say that it is:

- separated *if for all $a, b \in E$,*

$$a \sim b = 1 \textit{ implies } a = b. \tag{E9}$$

- residuated *if for all $a, b, c \in E$,*

$$(a \otimes b) \wedge c = a \otimes b \textit{ iff } a \wedge ((b \wedge c) \sim b) = a. \tag{E10}$$

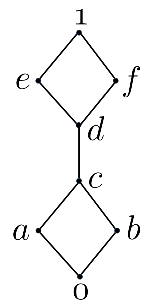

Figure 1: Eight elements good EQ-algebra

- good *if for all $a \in E$,*

$$a \sim 1 = a. \tag{E11}$$

- lattice-ordered *EQ-algebra if the underlying $\wedge$-semilattice is a lattice.*

- lattice EQ-algebra *(ℓEQ-algebra) if it is lattice-ordered in which the following substitution axiom holds, for all $a, b, c, d \in E$:*

$$((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c). \tag{E12}$$

Note that an EQ-algebra can be lattice-ordered but not ℓEQ-algebra. An EQ-algebra $\mathcal{E}$ is *complete* if it is a complete $\wedge$-semilattice. A complete EQ-algebra is a complete lattice-ordered (see, e.g. [4]). Every finite EQ-algebra is lattice-ordered. Clearly, (E10) can be written in a classical way as $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

Example 1
Example of a finite non-trivial good EQ-algebra is the following: its (semi)lattice structure is in Figure 1. Multiplication and fuzzy equality are defined as follows:

$\otimes$	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	0	0	0	0	0	b
c	0	0	0	0	0	0	a	c
d	0	0	0	0	d	d	d	d
e	0	0	0	0	d	e	d	e
f	0	0	0	0	d	d	d	f
1	0	a	b	c	d	e	f	1

$\sim$	**0**	**a**	**b**	**c**	**d**	**e**	**f**	**1**
0	1	e	f	d	c	a	b	0
a	e	1	d	f	c	a	c	a
b	f	d	1	e	c	c	b	b
c	d	f	e	1	c	c	c	c
d	c	c	c	c	1	f	e	d
e	a	a	c	c	f	1	d	e
f	b	c	b	c	e	d	1	f
1	0	a	b	c	d	e	f	1

Note that the multiplication in the above example is not commutative since $c \otimes f = a$ but $f \otimes c = 0$. Moreover, this algebra is non-residuated since, e.g., $0 = a \otimes f \leq b$, but $a \not\leq f \to b = b$.

Example 2 ([8])
Let $E = [0,1]$ and define a multiplication $\otimes$ and fuzzy equality $\sim$ on E as follows:

$$a \otimes b = \begin{cases} 0, & 2a + b \leq 1 \\ \min\{a, b\}, & 2a + b > 1 \end{cases}, \; a \sim c = \begin{cases} 1, & a = c \\ \max(\frac{1}{2} - a, c), & a > c \\ \max(\frac{1}{2} - c, a), & a < c \end{cases}$$

$\otimes$ is isotone monoidal operation on $[0,1]$, but it is neither commutative nor continuous but left-continuous. Hence, $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ is a good EQ-algebra. This algebra is non-residuated.

There are many other examples of EQ-algebras including linearly ordered ones (see [8, 23]).

2.2. Properties of EQ-algebras

Lemma 1 ([8])
Let $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ be an EQ-algebra. Then the following properties are provable for all a, b, c, d in E:

(EQ1) $a \sim b = b \sim a$, *(symmetry)*

(EQ2) $(a \sim b) \otimes (b \sim c) \leq (a \sim c)$, *(transitivity)*

(EQ3) $a \otimes b \leq a \wedge b \leq a,\ b$ and $b \otimes a \leq a \wedge b \leq a,\ b$,

(EQ4) $a \sim d \leq (a \wedge b) \sim (d \wedge b)$,

(EQ5) $a \sim b \leq a \to b$ and $a \to a = 1$, *(i.e. $\to$ is reflexive)*

(EQ6) $(a \to b) \otimes (b \to c) \leq a \to c$ and $(b \to c) \otimes (a \to b) \leq a \to c$, *(transitivity of implication)*

(EQ7) $(a \to b) \otimes (b \to a) \leq a \sim b \leq (a \to b) \wedge (b \to a)$. *If $\mathcal{E}$ is linearly ordered then both inequalities can be replaced by equalities,*

(EQ8) If $a \leq b$ then $a \to b = 1$, $a \sim b = b \to a$, $c \to a \leq c \to b$ and $b \to c \leq a \to c$,

(EQ9) $b \leq a \to b$,

(EQ10) $a \sim d \leq (a \sim c) \sim (d \sim c)$,

(EQ11) $a \to d \leq (b \to a) \to (b \to d)$,

(EQ12) $b \to a \leq (a \to d) \to (b \to d)$,

(EQ13) $a \to b \leq (a \wedge c) \to (b \wedge c)$,

(EQ14) $a \to b = a \to (a \wedge b)$.

By Lemma 1 (EQ5), (EQ6) and (EQ7), the fuzzy implication $\to$ is a fuzzy ordering w.r.t. the fuzzy equality $\sim$ (this notion was studied extensively by Bodenhofer [5]). As mentioned in [23], we can regard an EQ-algebra as a set endowed with a classical partial order $\leq$ (and corresponding equality $=$) and a top element 1, and a fuzzy equality $\sim$ together with a fuzzy ordering $\to$.

Theorem 1 ([8])
Let $\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1} \rangle$ be a semicopula-based EQ-algebra. Define the reverse $\bar{\otimes}$ of $\otimes$ by $a \bar{\otimes} b = b \otimes a$. Then $\bar{\mathcal{E}} = \langle E, \wedge, \bar{\otimes}, \sim, \mathbf{1} \rangle$ is a semicopula-based EQ-algebra.

2.3. Properties of special EQ-algebras

Proposition 1 ([8])
The following statements are equivalent:

(i) An EQ-algebra $\mathcal{E}$ is separated.

(ii) $a \leq b$ iff $a \to b = 1$ for all $a, b \in E$.

This means that the implication operation $\to$ in a separated EQ-algebra precisely reflects the ordering $\leq$ and so, the multiplication $\otimes$ is $\to$-isotone in it.

Proposition 2 ([8])
The following statements are equivalent:

(i) An EQ-algebra $\mathcal{E}$ is good.

(ii) $1 \to b = b$ for all $b \in E$.

The $\{\to, 1\}$-reducts[3] of good EQ-algebras are *BCK-algebras* (for the definitions and basic properties of BCK-algebras see [13, 19, 26, 27, 29]). Actually, we have the following theorem from [8]:

[3]Given an algebra $\langle E, F \rangle$, where F is the set of operations on E, and $F' \subseteq F$. Then the algebra $\langle E, F' \rangle$ is called the F'-reduct of $\langle E, F \rangle$. The subalgebras of $\langle E, F' \rangle$ are then referred to as F'-subreducts of $\langle E, F \rangle$.

Theorem 2
The $\{\wedge, \to, 1\}$*-reducts of good EQ-algebras are* BCK-meet-semilattices.

In consequence, all the properties concerning BCK-algbras are also the properties of the $\{\to, 1\}$-reducts of good EQ-algebras. Below, we list some properties from [8] that will be used in the paper:

Lemma 2 ([8])
Let $\mathcal{E}$ *be a good EQ-algebra. For all* $a, b, c \in E$*, it holds that*

(a) $\mathcal{E}$ *is separated and axiom (E8) is provable from the other EQ-axioms.*

(b) $a \leq (a \sim b) \sim b$.

(c) $a \leq (a \to b) \to b$.

(d) $a \leq b \to c$ *iff* $b \leq a \to c$

(e) $a \to (b \to c) \leq (a \otimes b) \to c$ *and* $a \to (b \to c) \leq (b \otimes a) \to c$.

(f) $a \to (b \to c) = b \to (a \to c)$ *(Exchange principle (EP)).*

(g) For all indexed families $\{a_i\}$ *in* E*, provided that* $\{a_i\}$ *has supremum in* E*, we have*
$$\bigvee_i a_i \to c = \bigwedge_i (a_i \to c).$$

(h) $(a \sim b) \otimes a \leq a \wedge b$ *and* $a \otimes (a \sim b) \leq a \wedge b$.

(i) $(a \to b) \otimes a \leq a \wedge b$ *and* $a \otimes (a \to b) \leq a \wedge b$.

(j) $a \leq b \to c$ *implies* $a \otimes b \leq c$ *and* $b \otimes a \leq c$.

Lemma 3 ([23])
The following holds in every complete EQ-algebra:

(i) $a \to \bigwedge_{i \in I} b_i \leq \bigwedge_{i \in I} (a \to b_i)$.

(ii) $\bigvee_{i \in I} (a_i \to b) \leq \bigwedge_{i \in I} a_i \to b$.

Theorem 3 ([8])
If $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ *is a residuated EQ-algebra, then*

(i) $\mathcal{E}$ *is a residuated and commutative EQ-algebra.*

(ii) If $\mathcal{E}$ *is also lattice-ordered, then* $\mathcal{E}' = \langle E, \wedge, \vee, \otimes, \to, 1\rangle$ *is a commutative residuated lattice with* $\to$ *defined by* $a \to b = (a \wedge b) \sim a$.

Notice that, by Theorem 3, adding the adjunction condition to EQ-algebra leads to a commutative residuated lattice and not to a non-commutative one. This means that EQ-algebras generalize commutative residuated lattices.

Proposition 3 ([8])
The following statements are equivalent in an EQ-algebra $\mathcal{E}$.

(i) $\mathcal{E}$ *is residuated.*

(ii) $\mathcal{E}$ *is good and satisfies*

$$a \to b \leq (a \otimes c) \to (b \otimes c),\ \forall a, b, c \in E. \tag{4}$$

(iii) $\mathcal{E}$ *is good and satisfies*

$$a \leq b \to (a \otimes b),\ \forall a, b \in E. \tag{5}$$

Proposition 4 ([8])
The following statements are equivalent in a lattice-ordered EQ-algebra $\mathcal{E}$.

(i) $\mathcal{E}$ *is ℓEQ-algebra.*

(ii) $\mathcal{E}$ *satisfies, for all a, b, c in E*

$$a \sim b \leq (a \vee c) \sim (b \vee c). \tag{6}$$

Lemma 4 ([8])
Let $\mathcal{E}$ be a ℓEQ-algebra. For all $a, b, c \in E$, it holds that

(i) $a \to b = (a \vee b) \to b = (a \vee b) \sim b$.

(ii) $a \to b \leq (a \vee c) \to (b \vee c)$.

3. Prelinear EQ-algebras

Definition 3
An EQ-algebra $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ is said to be prelinear if for all $a, b \in E$, 1 is the unique upper bound in E of the set $\{(a \to b), (b \to a)\}$.

Note that the prelinearity does not necessitate the presence of a join operator in E[4]. However, in the following, we will show that every prelinear and good EQ-algebra is a lattice-ordered whereby the join operation is definable in terms of the meet $\wedge$ and the implication $\to$ operations (see Theorem 4).

The EQ-algebra in Example 1 is prelinear. Also, linearly ordered EQ-algebras and their direct products are prelinear.

Let us put

$$a \leftrightarrow b = (a \to b) \wedge (b \to a), \tag{7}$$

$$a \overset{\circ}{\leftrightarrow} b = (a \to b) \otimes (b \to a). \tag{8}$$

[4]This approach is well known in literature, see e.g., A. Abdel-Hamid, Morsi [1] where the authors established a representation theorem of prelinear residuated algebras, in which the lattice structure is not assummed.

Lemma 5

Let $\mathcal{E}$ be a prelinear and separated EQ-algebra. Then, for all $a, b, c, d \in E$, it holds that

(i) $a \leftrightarrow b = a \sim b$.

(ii) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.

(iii) $(a \sim b) \wedge (c \sim d) \leq (a \wedge c) \sim (b \wedge d)$.

PROOF: (i) By Lemma 1 (EQ10), we have

$(a \wedge b) \sim a \leq ((a \wedge b) \sim b) \sim (a \sim b)$. Hence, by (3) and the order properties of $\to$, $a \to b \leq (b \to a) \to (a \sim b) \leq (a \leftrightarrow b) \to (a \sim b)$. Similarly, $b \to a \leq (a \leftrightarrow b) \to (a \sim b)$. Hence by prelinearity, $(a \leftrightarrow b) \to (a \sim b) = 1$; that is (by separation and Proposition 1 (ii)) $a \leftrightarrow b \leq a \sim b$. Thus, by Lemma 1 (EQ7), the equality holds.

(ii) By Lemma 1 (EQ14) and the order properties of $\to$ (see Lemma 1 (EQ11) and (EQ12)), we get

$b \to c = b \to (b \wedge c) \leq (a \to b) \to (a \to (b \wedge c)) \leq ((a \to b) \wedge (a \to c)) \to (a \to (b \wedge c))$. Similarly, $c \to b \leq ((a \to b) \wedge (a \to c)) \to (a \to (b \wedge c))$. Hence by prelinearity, $((a \to b) \wedge (a \to c)) \to (a \to (b \wedge c)) = 1$; that is (by separation and Proposition 1 (ii)) $(a \to b) \wedge (a \to c) \leq a \to (b \wedge c)$. The opposite inequality follows from Lemma 3 (i).

(iii) By item (i) and (7), we have

$(a \sim b) \wedge (c \sim d) = (a \leftrightarrow b) \wedge (c \leftrightarrow d) = ((a \to b) \wedge (b \to a)) \wedge ((c \to d) \wedge (d \to c)) = ((a \to b) \wedge (c \to d)) \wedge ((b \to a) \wedge (d \to c))$.

Hence, by the order properties of $\to$, we get

$(a \sim b) \wedge (c \sim d) \leq (((a \wedge c) \to b) \wedge ((a \wedge c) \to d)) \wedge (((b \wedge d) \to a) \wedge ((b \wedge d) \to c))$.

Thus by item (ii), we get

$(a \sim b) \wedge (c \sim d) \leq ((a \wedge c) \to (b \wedge d)) \wedge ((b \wedge d) \to (a \wedge c)) = (a \wedge c) \leftrightarrow (b \wedge d)$ (by (7)). Hence by item (i), we get the result. $\square$

Lemma 6

Let $\mathcal{E}$ be a prelinear and separated ℓEQ-algebra. Then, for all $a, b, c \in E$, it holds that

(i) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.

(ii) $a \sim b = (a \vee b) \to (a \wedge b)$.

PROOF: (i) By Lemma 4 (i) and the order properties of $\to$ (Lemma 1 (EQ12)), we get

$a \to b = (a \vee b) \to b \leq (b \to c) \to ((a \vee b) \to c) \leq ((a \to c) \wedge (b \to c)) \to ((a \vee b) \to c)$. Similarly, $b \to a \leq ((a \to c) \wedge (b \to c)) \to ((a \vee b) \to c)$. Hence by prelinearity, $((a \to c) \wedge (b \to c)) \to ((a \vee b) \to c) = 1$; that is (by separation and Proposition 1 (ii)) $(((a \to c) \wedge (b \to c)) \leq ((a \vee b) \to c)$.

On the other hand, by the order properties of $\rightarrow$, we have $(a \vee b) \rightarrow c \leq a \rightarrow c$ and $(a \vee b) \rightarrow c \leq b \rightarrow c$. Hence, $((a \vee b) \rightarrow c) \leq (((a \rightarrow c) \wedge (b \rightarrow c))$.

(ii) By Lemma 5 (ii) and Lemma 4 (i), we have

$(a \vee b) \rightarrow (a \wedge b) = ((a \vee b) \rightarrow a) \wedge ((a \vee b) \rightarrow b) = (a \rightarrow b) \wedge (b \rightarrow a) = a \leftrightarrow b$. Hence by Lemma 5 (i), we get the result. □

Theorem 4
Every prelinear and good EQ-algebra $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ is a prelinear and good ℓEQ-algebra, whereby the join operation is given by

$$a \vee b = ((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a), \qquad a, b \in E \tag{9}$$

PROOF: It is well known that every prelinear residuated $\wedge$-semilattice (see [18]) is a lattice, whereby the join operation is given by (9). The machinery employed in the proof consists of Lemma 2 (c) and Lemma 1 (EQ9) together with the order properties of $\rightarrow$. So, the proof is valid in the present setting and applies verbatim here for prelinear and good EQ-algebras (see also [1, Lemma 5.1]). For the reader convenience, we will supply the proof:
Denote $((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a)$ by γ. It follows form Lemma 2 (c) and Lemma 1 (EQ9) that γ is an upper bound of the set $\{a, b\}$. Let δ be another upper bound (i.e. $a, b \leq \delta$). Then, by Lemma 2 (c) and the order properties of $\rightarrow$, $a \rightarrow b \leq ((a \rightarrow b) \rightarrow b) \rightarrow b \leq \gamma \rightarrow \delta$. Similarly, $b \rightarrow a \leq \gamma \rightarrow \delta$. Hence by prelinearity, $\gamma \rightarrow \delta = 1$; that is $\gamma \leq \delta$. This shows that γ is the least upper bound $a \vee b$ of a and b and hence $\mathcal{E}$ is a lattice-ordered, whereby the join operation is given by (9).
It remains only to show that $\mathcal{E}$ satisfies (E12). By Lemma 4 (ii), we get $a \rightarrow b \leq (a \vee c) \rightarrow (b \vee c)$ and $b \rightarrow a \leq (b \vee c) \rightarrow (a \vee c)$. Hence, $a \leftrightarrow b \leq (a \vee c) \leftrightarrow (b \vee c)$. Thus, by Lemma 5 (i), we obtain $a \sim b \leq (a \vee c) \sim (b \vee c)$. Hence, by Proposition 4, $\mathcal{E}$ is a ℓEQ-algebra. □

In consequence, all properties of good ℓEQ-algebras are also properties of prelinear and good EQ-algebras, see [8] and [23] for the properties of good ℓEQ-algebras. Note that equation (9) is known to hold in *MTL-algebras* [9].

The use of Theorem 4 pervades most proofs in this work. However, throughout this article, we will use it without explicit mention.

Recall that a prelinear commutative residuated lattice is called in [20] a *prelinear hoop* (or, *basic semihoop* [10]); that is, an *MTL-algebra* [9] that need not have a zero element.

Theorem 5
Let $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ be a prelinear residuated EQ-algebra. Then $\mathcal{E}' = \langle E, \wedge, \vee, \otimes, \rightarrow, 1\rangle$ is a prelinear commutative residuated lattice (i.e. basic semihoop), whereby the join operation is given by (9) and $\rightarrow$ is defined by $a \rightarrow b = (a \wedge b) \sim a$. Moreover, if $\mathcal{E}$ has a bottom element then $\mathcal{E}'$ is an MTL-algebra.

PROOF: By Proposition 3, $\mathcal{E}$ is good. Hence, by the assumptions and Theorem 4, $\mathcal{E}$ is a prelinear and residuated ℓEQ-algebra. The rest is evident by Theorem

3. $\square$

Proposition 5
Let $\mathcal{E}$ be a prelinear and good EQ-algebra. Then, the following are equivalent:

(i) $a \vee b = 1$,

(ii) $a \to b = b$ *and* $b \to a = a$.

PROOF: First recall that prelinear and good EQ-algebra is an ℓEQ-algebra (Theorem 4). Now, assume (i). By Lemma 4 (i) and by goodness, we get
$a = 1 \to a = (a \vee b) \to a = b \to a$ and $b = 1 \to b = (a \vee b) \to b = a \to b$.
Hence (i) implies (ii). The converse follows directly by the prelinearity. $\square$

Proposition 6
Let $\mathcal{E}$ be a prelinear and good EQ-algebra. Then, the following are equivalent:

(i) $a \vee b = 1$ *implies* $a \otimes b = a \wedge b$.

(ii) $a \overset{\circ}{\leftrightarrow} b = a \sim b$.

PROOF: (i) implies (ii): Direct by prelinearity and Lemma 5 (i).

(ii) implies (i): Assume (ii) holds and let $a \vee b = 1$, then by Proposition 5, $a \to b = b$ and $b \to a = a$. Hence, $a \otimes b = (b \to a) \otimes (a \to b) = a \overset{\circ}{\leftrightarrow} b = a \sim b = (b \to a) \wedge (a \to b) = a \wedge b$ (by Lemma 5 (i) and Proposition 5). $\square$

Note that, in general, in a prelinear and good (commutative) EQ-algebra $a \overset{\circ}{\leftrightarrow} b \neq a \sim b$ (see Example 3). But, this identity always holds for all linearly ordered EQ-algebras. This shows that prelinearity alone does not characterize the representable class of all good EQ-algebras. However, as we will see in the rest of this section, a lot of properties analogous to the ones obtained for prelinear residuated lattice in [1, 18], BL in [16] and MTL in [9] still hold in prelinear and separated lattice EQ-algebras. We devote the rest of this section to proofs of some of those new properties. We also show which of them is equivalent to prelinearity.

Lemma 7
A lattice-ordered and separated EQ-algebra $\mathcal{E}$ is prelinear if and only if the following identity holds, for all $a, b, c \in E$:

$$(a \wedge b) \to c = (a \to c) \vee (b \to c). \tag{10}$$

PROOF: If $\mathcal{E}$ is prelinear then, by Lemma 1 (EQ12) and (EQ14), we get
$a \to b = a \to (a \wedge b) \leq ((a \wedge b) \to c) \to (a \to c) \leq ((a \wedge b) \to c) \to ((a \to c) \vee (b \to c))$. Similarly, $b \to a \leq ((a \wedge b) \to c) \to ((a \to c) \vee (b \to c))$. Hence by prelinearity, $((a \wedge b) \to c) \to ((a \to c) \vee (b \to c)) = 1$; that is (by separation

and Proposition 1 (ii)) $(a \wedge b) \to c \leq (a \to c) \vee (b \to c)$. The other inequality follows from Lemma 3 (ii). Hence, prelinearity implies that $\mathcal{E}$ satisfies (10).

Conversely, assume that $\mathcal{E}$ satisfies (10). Then for all $a, b \in E$: $1 = (a \wedge b) \to (a \wedge b) = (a \to (a \wedge b)) \vee (b \to (a \wedge b)) = (a \to b) \vee (b \to a)$; that is $\mathcal{E}$ is prelinear. □

Proposition 7

Let $\mathcal{E}$ be a separated ℓEQ-algebra. Then, the following statements are equivalent, for all $a, b, c \in E$:

(i) $\mathcal{E}$ *is prelinear.*

(ii) $a \to (b \vee c) = (a \to b) \vee (a \to c)$.

(iii) $a \to c \leq (a \to b) \vee (b \to c)$.

PROOF: (i) implies (ii): Assume $\mathcal{E}$ is prelinear. Then, by Lemma 4 (i) and Lemma 1 (EQ11), we get

$b \to c = (b \vee c) \to c \leq (a \to (b \vee c)) \to (a \to c) \leq (a \to (b \vee c)) \to ((a \to c) \vee (b \to c))$. Similarly, $c \to b \leq (a \to (b \vee c)) \to ((a \to c) \vee (b \to c))$. Hence by prelinearity, $(a \to (b \vee c)) \to ((a \to c) \vee (b \to c)) = 1$; that is (by separation) $a \to (b \vee c) \leq (a \to b) \vee (a \to c)$. The opposite inequality follows from the monotonicity of $\to$ in the right argument.

(ii) implies (iii): Assume (ii). Then, by Lemma 4 (ii), $a \to c \leq (b \vee a) \to (b \vee c) = ((b \vee a) \to b) \vee ((b \vee a) \to c) \leq (a \to b) \vee (b \to c)$ (by the order properties of $\to$).

(iii) implies (i): Assume (iii). Then for all $b, c \in E$: $1 = c \to c \leq (c \to b) \vee (b \to c)$; that is $\mathcal{E}$ is prelinear. □

Lemma 8

A prelinear and separated ℓEQ-algebra is distributive*; that is, for all $a, b, c \in E$ it holds that: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.*

PROOF: By Lemma 4 (i) and Lemma 1 (EQ13), we have

$b \to c = (b \vee c) \to c \leq (a \wedge (b \vee c)) \to (a \wedge c)$.

Similarly, $c \to b = (b \vee c) \to b \leq (a \wedge (b \vee c)) \to (a \wedge b)$. Hence, by prelinearity and Lemma 7 (ii), we get

$1 = (b \to c) \vee (c \to b) \leq ((a \wedge (b \vee c)) \to (a \wedge c)) \vee ((a \wedge (b \vee c)) \to (a \wedge b)) = (a \wedge (b \vee c)) \to ((a \wedge b) \vee (a \wedge c))$.

Hence, $(a \wedge (b \vee c)) \to ((a \wedge b) \vee (a \wedge c)) = 1$; that is $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$.

This proves distributivity, because the opposite inequality is always valid. □

Note that the dual of the identity in Lemma 8 (i.e., $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$) holds and both of the two identities are equivalent with each other.
The relation of EQ-algebras to residuated lattices is quite intricate and it seems

that the former opened the door to another look at the latter. When considering implication only, $\{\to, 1\}$-reducts of good EQ-algebras are *BCK-algebras*. It is commonly known that BCK-algebras are exactly the $\{\to, 1\}$-subreducts of commutative integral residuated lattices (see [12, 25, 28]), and so, residuated lattices are " hidden" inside. Moreover, $\{\wedge, \to, 1\}$-reducts of good EQ-algebras are *BCK-meet-semilattices*. It should be emphasized here that BCK-meet-semilattices (respectively, BCK-lattices) are not the corresponding subreducts of integral residuated lattices. As a matter of fact, BCK-lattice can be embedded into the $\{\wedge, \to, 1\}$-reduct of a commutative residuated lattice if and only if it satisfies the identity in Lemma 5 (ii), which is easily seen to hold in residuated lattices. The $\{\wedge, \vee, \to, 1\}$-reduct of prelinear good EQ-algebra is a *BCK-lattice* and satisfies this identity, and so it can be embedded into the $\{\wedge, \vee, \to, 1\}$-reduct of a commutative residuated lattice. We will focus more closely on this relation in subsequent papers.

On the other hand, it is known that representable commutative residuated lattices and representable BCK-algebras are characterized by the prelinearity. It follows that a $\{\wedge, \vee, \to, 1\}$-reducts of prelinear and good EQ-algebra inherits several lattice-related properties from products of BCK-chains (or from product of $\{\wedge, \vee, \to, 1\}$-reducts of residuated chains). For example, it is well known that the prelinearity is equivalent to the following M. Pałasinski's [26] identity (11). Accordingly, we have the following lemma and there is no need to prove it since it is already known fact on BCK-algebras (see [26]):

Lemma 9

A good EQ-algebra $\mathcal{E}$ is prelinear if and only if the following inequality holds for all $a, b, c \in E$:

$$(a \to b) \to c \leq ((b \to a) \to c) \to c. \tag{11}$$

Inequality (11) has been chosen by Hájek [16] as the axiom of prelinearity in his axiomatization of BL-algebras, apparently because it is free from lattice operations.

We end this section by two examples which show that the prelinearity alone does not characterize the representable class of all good (commutative) EQ-algebras.

Example 3

Let E be the bounded lattice $\{0, \alpha, \beta, \gamma, 1\}$ with the partial order $\leq$ defined by: $0 < \alpha < \beta < 1$ and $0 < \alpha < \gamma < 1$, whereas β and γ are non-comparable. The following multiplication and the fuzzy equality define a prelinear and good EQ-algebra in which the identity $a \overset{\circ}{\leftrightarrow} b = a \sim b$ does not hold for all $a, b \in E$, since, e.g., $\alpha = \beta \sim \gamma \neq (\beta \to \gamma) \otimes (\gamma \to \beta) = \gamma \otimes \beta = 0$.

$\otimes$	$\mathbf{0}$	α	β	γ	$\mathbf{1}$
$\mathbf{0}$	0	0	0	0	0
α	0	0	0	α	α
β	0	α	β	α	β
γ	0	0	0	γ	γ
$\mathbf{1}$	0	α	β	γ	1

$\sim$	$\mathbf{0}$	α	β	γ	$\mathbf{1}$
$\mathbf{0}$	1	0	0	0	0
α	0	1	α	α	α
β	0	α	1	α	β
γ	0	α	α	1	γ
$\mathbf{1}$	0	α	β	γ	1

$\rightarrow$	$\mathbf{0}$	α	β	γ	$\mathbf{1}$
$\mathbf{0}$	1	1	1	1	1
α	0	1	1	1	1
β	0	α	1	γ	1
γ	0	α	β	1	1
$\mathbf{1}$	0	α	β	γ	1

Example 4
Let $\mathcal{E}$ be a finite prelinear and good commutative EQ-algebra whose lattice structure, fuzzy equality and implication as in Example 3 and whose commutative multiplication $\otimes$ is defined as follows:

$\otimes$	$\mathbf{0}$	α	β	γ	$\mathbf{1}$
$\mathbf{0}$	0	0	0	0	0
α	0	0	α	0	α
β	0	α	β	α	β
γ	0	0	α	γ	γ
$\mathbf{1}$	0	α	β	γ	1

It is easy to show that, for all $x, y, z \in E$, the identity $(y \rightarrow z) \vee (x \rightarrow ((z \rightarrow y) \otimes x)) = 1$ always holds for all linearly ordered EQ-algebras. However, $\mathcal{E}$ fails to satisfy it when $x = \alpha$, $y = \gamma$, and $z = \beta$.

4. Prefilters, filters and congruences

EQ-algebras behave differently than residuated lattices, as is illustrated (see [8]) by the fact that $a \rightarrow b = 1$ does not imply that $a \leq b$. Not surprisingly then, the study of filters requires proper care, especially related to the behavior of $\sim$ w.r.t. $\otimes$. So, in [8], the study of filters was restricted to EQ-algebras which satisfy the separation axiom (E9) (recall that, in separated EQ-algebras, $a \rightarrow b = 1$ iff $a \leq b$).

Below, we recall some definitions and results from [8] that will be used in the paper.

Theorem 6 ([8])
Let θ be a congruence on an (good) EQ-algebra $\mathcal{E}$. Then the factor algebra $\mathcal{E}/\theta$ is an (good, and hence separated) EQ-algebra and the mapping $q : E \longrightarrow E/\theta$ defined by $q(a) = [a]_\theta$ is a homomorphism.

Note that if $\mathcal{E}$ is a separated EQ-algebra then the algebra $\mathcal{E}/\theta$ is not, in general, separated. Given a separated EQ-algebra $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$, we shall say that $\theta \in \mathbf{Con}(\mathcal{E})$ is a *relative congruence* of $\mathcal{E}$ if the quotient algebra $\mathcal{E}/\theta$ is a a separated EQ-algebra. Note that the trivial congruence is a relative congruence.

Definition 4 ([8])
Let $\mathcal{E} = \langle E, \wedge, \otimes, \sim, 1\rangle$ be a separated EQ-algebra. A subset $F \subseteq E$ is called a prefilter *of $\mathcal{E}$ if for all $a, b \in E$:*

(i) $1 \in F$.

(ii) *if $a, a \to b \in F$, then $b \in F$.*

A prefilter F is said to be filter *if, for all $a, b, c \in E$, $a \to b \in F$ implies $(a \otimes c) \to (b \otimes c) \in F$ and $(c \otimes a) \to (c \otimes b) \in F$.*

As usual, a prefilter F is *proper* if $F \neq E$. If $0 \in E$ then a prefilter $F \subset E$ is proper iff $0 \notin F$.

The singleton $\{1\}$ is a filter in any separated EQ-algebra, and is contained in any other filter.

Lemma 10 ([8])
Let F be a prefilter of a separated EQ-algebra $\mathcal{E}$. For all $a, b \in E$ it holds that

(i) If $a \in F$ and $a \leq b$ then $b \in F$.

(ii) If $a, a \sim b \in F$, then $b \in F$.

(iii) If $a, b \in F$ then $a \wedge b \in F$.

Moreover, if F is a filter, then for all $a, b \in E$ it holds that

(iv) If $a, b \in F$ then $a \otimes b \in F$.

(v) $a \sim b \in F$ iff $a \leftrightarrow b \in F$ iff "$a \to b \in F$ and $b \to a \in F$" iff $a \overset{\circ}{\leftrightarrow} b \in F$.

Remark 1
Let F be a prefilter of a prelinear and separated EQ-algebra $\mathcal{E}$, then (by Lemma 5 (i) and Lemma 10 (iii))

$$a \sim b \in F \text{ iff } a \leftrightarrow b \in F \text{ iff "} a \to b \in F \text{ and } b \to a \in F\text{"}.$$

Given a prefilter $F \subseteq E$, as usual, the following relation on E is an equivalence relation, but it is not congruence:

$$a \approx_F b \text{ iff } a \sim b \in F \tag{12}$$

It has been shown that (see [8]) if F is a prefilter of a separated (ℓ)EQ-algebra $\mathcal{E}$, then all the operations of $\mathcal{E}$ except the multiplication are compatible with the equivalence relation $\approx_F$ given by (12). That is $a \approx_F b$ and $a' \approx_F b'$ imply $(a \wedge a') \approx_F (b \wedge b')$, $(a \vee a') \approx_F (b \vee b')$ and $(a \sim a') \approx_F (b \sim b')$. If F is a filter, then $\approx_F$ is a congruence. Moreover, we have the following theorem from [8]:

Theorem 7 ([8])
Let F be a filter of a separated (ℓ)EQ-algebra $\mathcal{E}$. Then the relation $\approx_F$ given by (12) is a relative congruence of $\mathcal{E}$.

We shall denote by $[a]_F$ the equivalence class of $a \in E$ with respect to $\approx_F$ and by E/F the quotient set associated with $\approx_F$.

Let F be a prefilter of a separated ℓEQ-algebra $\mathcal{E}$. Then, one can define on E/F the binary operation $\wedge_F$ as $[x]_F \wedge_F [y]_F = [x \wedge y]_F$, and similarly for the other operations $\sim_F$, $\vee_F$ and $\to_F$. Also, one can define on E/F a binary relation $\leq_F$ as follows:

$$[a]_F \leq_F [b]_F \text{ iff } [a]_F \wedge_F [b]_F = [a]_F \tag{13}$$

Then we have the following result:

Theorem 8
Let F be a prefilter of a prelinear and separated ℓEQ-algebra $\mathcal{E}$. Then $\mathcal{E}/F = \langle E/F, \wedge_F, \vee_F, 1_F\rangle$ is a distributive lattice with top element $1_F = [1]_F = F$ and $f : a \mapsto [a]_F$ is a homomorphism of $\mathcal{E}$ onto $\mathcal{E}/F$. Moreover, the partial order $\leq_F$ given by (13) satisfies:

$$[a]_F \leq_F [b]_F \text{ iff } a \to b \in F \text{ iff } [a]_F \to_F [b]_F = [1]_F. \tag{14}$$

PROOF: It is already proved that, see [8], $\langle E/F, \wedge_F, 1_F\rangle$ is a meet-semilattice with top element $1_F = F$ and the binary relation $\leq_F$ given by (13) is a partial order on E/F and satisfies (14). The rest follows directly from Lemma 8. $\square$

Lemma 11
Let $\mathcal{E}$ be a separated EQ-algebra. For any relative congruence θ of $\mathcal{E}$, we have

(i) $F = [1]_\theta = \{a \in E : a\theta 1\}$ *is a filter of* $\mathcal{E}$.

(ii) $a\theta b$ *iff* $(a \sim b)\theta 1$ *iff* $(a \to b)\theta 1$ *and* $(b \to a)\theta 1$ *iff* $(a \leftrightarrow b)\theta 1$ *iff* $(a \overset{\circ}{\leftrightarrow} b)\theta 1$.

(iii) $[1]_\theta = \{1\}$ *iff* θ *is the trivial congruence.*

PROOF: (i) It is obvious that $1 \in [1]_\theta$. If $a, a \to b \in F$ then $[a]_\theta = [1]_\theta$ and $[a \to b]_\theta = [1]_\theta \to [b]_\theta = [1]_\theta$; that is $[1]_\theta \sim [b]_\theta = [1]_\theta$. Hence, $[1]_\theta = [b]_\theta$ (since θ is a relative congruence); i.e. $b \in F$. Now assume $a \to b \in F$. Hence $[a]_\theta \to [b]_\theta = [1]_\theta$. Thus, since θ is a relative congruence, $[a]_\theta \leq [b]_\theta$ and hence $[a \otimes c]_\theta \to [b \otimes c]_\theta = [1]_\theta$; that is $(a \otimes c) \to (b \otimes c) \in F$. Similarly, $(c \otimes a) \to (c \otimes b) \in F$.

(ii) From $a\theta b$ one infers $(a \sim a)\theta(a \sim b)$; that is $(a \sim b)\theta 1$. Conversely, $(a \sim b)\theta 1$ implies $[a \sim b]_\theta = [1]_\theta$; that is $[a]_\theta \sim [b]_\theta = [1]_\theta$. Thus, since θ is a relative congruence, $[a]_\theta = [b]_\theta$, i.e. $a\theta b$. The rest follows by item (i) and Lemma 10 (v).

(iii) First of all note that the trivial congruence is a relative congruence. By item (i), $[1]_\theta$ is a filter. Suppose that $[1]_\theta = \{1\}$. If $a\theta b$ then $(a \sim b)\theta 1$ (by item (ii)), hence $a \sim b = 1$ so $a = b$ (since $\mathcal{E}$ is separated). Thus, θ is the trivial congruence. □

Corollary 1
If θ and ϕ are relative congruences of a separated EQ-algebra, then $[1]_\theta = [1]_\phi$ implies $\theta = \phi$.

PROOF: Assume $[1]_\theta = [1]_\phi$. Hence, by Lemma 11 (ii), we get $a\theta b$ iff $(a \sim b)\theta 1$ iff $(a \sim b)\phi 1$ iff $a\phi b$. □

The collection of all filters of a separated EQ-algebra $\mathcal{E}$ will be denoted by $\mathcal{F}(\mathcal{E})$. This is easily seen to be a lattice (which form a complete sublattice of the lattice of all prefilters, denoted by $\mathbf{P}\mathcal{F}(\mathcal{E})$), with meets given by intersections.

Theorem 9
For any separated EQ-algebra $\mathcal{E}$, the lattice $\mathcal{F}(\mathcal{E})$ of filters of $\mathcal{E}$ is isomorphic to the lattice of relative congruences of $\mathcal{E}$, via the mutually inverse maps $F \longmapsto \approx_F$ and $\theta \longmapsto [1]_\theta$.

PROOF: By Theorem 7, the relation $\approx_F$ given by (12) is a relative congruence relation on $\mathcal{E}$, and by Lemma 11 (i), $[1]_\theta$ is a filter of $\mathcal{E}$. Since the given maps are clearly order-preserving, it suffices to show they are inverses of each other, since it will then follow that they are lattice homorphisms. We have proved in Theorem 8 that $F = [1]_{\approx_F}$. To show that $\theta = \approx_{[1]_\theta}$, we let $F = [1]_\theta$ and observe that $[1]_{\approx_F} = F = [1]_\theta$. Hence, by Corollary 1, the result follows. □

Since, by Theorem 6, the factor algebra $\mathcal{E}/\theta$ of a good EQ-algebra is a good (and hence separated) EQ-algebra, any congruence is a relative congruence. Hence we have the following result as a corollary of Theorem 9:

Theorem 10
For any good EQ-algebra $\mathcal{E}$, the lattice $\mathcal{F}(\mathcal{E})$ of filters of $\mathcal{E}$ is isomorphic to the lattice $\mathbf{Con}(\mathcal{E})$ of congruences of $\mathcal{E}$, via the mutually inverse maps $F \longmapsto \approx_F$ and $\theta \longmapsto [1]_\theta$.

Definition 5
A prefilter F of a separated EQ-algebra $\mathcal{E}$ is said to be a prime prefilter *(or simply* prime*) if $\forall a, b \in E$ we find that $a \to b \in F$ or $b \to a \in F$.*

Note that if F is prime and Q is a prefilter such that $F \subseteq Q$, then Q is a prime prefilter.

Proposition 8
Let F be a prefilter of a prelinear and separated ℓEQ-algebra $\mathcal{E}$. Then the following are equivalent:

(i) *F is prime,*

(ii) *for each $a, b \in E$ such that $a \vee b \in F$, $a \in F$ or $b \in F$,*

(iii) *for each $a, b \in E$ such that $a \vee b = 1$, $a \in F$ or $b \in F$,*

(iv) *E/F is a chain, i.e. is linearly (totally) ordered by $\leq_F$.*

PROOF: *(i) entails (ii):* Assume F is a prime filter, and let $a \vee b \in F$. By prelinearity, $(a \to b) \vee (a \to b) = 1$. Since F is prime, then $a \to b \in F$, say, and hence by Lemma 4 (i), $(a \vee b) \to b = a \to b \in F$. Thus by Definition 4, $b \in F$.

(ii) entails (iii): $1 \in F$.

(iii) entails (iv) and (iv) entails (i): Conjoin (iii) with the prelinearity, we get $a \to b \in F$ or $b \to a \in F$; that is F is prime, this is equivalent (by (14)) to $[a]_F \leq_F [b]_F$ or $[b]_F \leq_F [a]_F$; that is E/F is a chain. □

The set-subtraction of set Y from set X will be denoted $X - Y$. Recall that an element b of a lattice is *meet-irreducible* if, for any finite and non-empty set X, $\bigwedge X = b$ implies $b \in X$. If this property holds for all non-empty sets X, we call b *completely meet-irreducible*. Let F be a prefilter of a separated EQ-algebra $\mathcal{E}$ and let $a \in E$. We say that F is *a-maximal* if $a \notin F$ but $a \in Q$ for any prefilter Q that properly contains F (denoted $F \subset Q$).

Lemma 12
Let F be a prime prefilter of a prelinear and separated ℓEQ-algebra $\mathcal{E}$. Then

(i) *$\{Q \in \mathbf{P}\mathcal{F}(\mathcal{E}) : F \subseteq Q\}$ is linearly ordered under inclusion.*

(ii) *F is meet-irreducible element in $\mathbf{P}\mathcal{F}(\mathcal{E})$.*

PROOF: (i) Let $F \subseteq Q, R$ and suppose that Q and R are incomparable. Then there exist $a \in Q - R$ and $b \in R - Q$. By Proposition 8, E/F is linearly ordered by $\leq_F$. Without loss of generality, suppose that $[a]_F \leq_F [b]_F$, i.e., $a \to b \in F$, hence $a \to b \in Q$. Since Q is a prefilter and $a \in Q$, we have $b \in Q$ as well, which is a contradiction.

(ii) Let $Q, R \in \mathbf{P}\mathcal{F}(\mathcal{E})$. If $Q \cap R = F$, then $F \subseteq Q, R$. So, by item (i) and without loss of generality, $Q \subseteq R$, hence $Q = F$. □

Lemma 13
Let $\mathcal{E}$ be a separated EQ-algebra. A prefilter F of $\mathcal{E}$ is a-maximal for some $a \in E - \{1\}$ iff F is completely meet-irreducible in $\mathbf{P}\mathcal{F}(\mathcal{E})$. Thus,

$$\bigcap\{F : F \text{ is completely meet-irreducible element in } \mathbf{P}\mathcal{F}(\mathcal{E})\} = \{1\},$$

hence

$$\bigcap\{F : F \text{ is meet-irreducible element in } \mathbf{P}\mathcal{F}(\mathcal{E})\} = \{1\}.$$

PROOF: Just note that $\{1\}$ is a prefilter of $\mathcal{E}$ and is contained in any other prefilter. Hence, the proof proceeds in a standard way (cf. [2, Lemma 2.6]). □

5. Representable good EQ-algebras

Recall that an EQ-algebra which is a subdirect product of those with underlying linear order is said to be *representable*. We devote this section to characterize the representable class of good EQ-algebras, along lines parallel to C. J. van Alten's characterization of representable integral residuated lattices [2].

As mentioned before, the $\{\rightarrow, 1\}$-reducts of good EQ-algebras are BCK-algebras. Moreover, our definition of (prime) prefiters on good EQ-algebras coincides with the definition of (prime) filters on BCK-algebras (called (linear) deductive filters in [14]). Accordingly, all the properties concerning the (prime) filters in BCK-algebras are also the properties of (prime) prefilters on good EQ-algebras, including Lemma 14 below (see [2, 14, 27] for the analogous results in BCK-algebras and residuated lattices).

Lemma 14
Let $\mathcal{E}$ be a good EQ-algebra. Then

- *(i) The lattice $\mathbf{P}\mathcal{F}(\mathcal{E})$ of all prefilters of $\mathcal{E}$ is a complete distributive lattice.*
- *(ii) Every meet-irreducible prefilter of $\mathcal{E}$ contains a minimal meet-irreducible prefilter.*
- *(iii) $\bigcap\{F : F$ is a minimal meet-irreducible element in $\mathbf{P}\mathcal{F}(\mathcal{E})\} = \{1\}$.*
- *(iv) If $\mathcal{E}$ is prelinear, then a prefilter F is prime iff F is meet-irreducible element in $\mathbf{P}\mathcal{F}(\mathcal{E})$.*

PROOF: (i) This is a well-known property of the deductive systems of BCK-algebras (see [27]).

(ii) Since $\mathbf{P}\mathcal{F}(\mathcal{E})$ is a complete distributive lattice, (ii) follows from a result by Alten (see [2, Lemma 2.7]) about complete distributive lattices.

(iii) follows from item (ii) and Lemma 13.

(iv) This is a well-known property of the prime deductive systems of BCK-algebras (see [14, Theorem 4.4]). □

Proposition 9

For a prefilter F of a good EQ-algebra $\mathcal{E}$, the following are equivalent.

(i) F *is a filter.*

(ii) *For all* $b, c \in E$, $b \in F$ *implies* $c \to (b \otimes c) \in F$ *and* $c \to (c \otimes b) \in F$.

(iii) *For all* $b, c, d \in E$, $b \in F$ *implies* $d \to (d \otimes (c \to (b \otimes c))) \in F$.

PROOF: (i) $\Longrightarrow$ (ii): Assume that F is a filter and let $b \in F$. Hence, since $\mathcal{E}$ is good, $b = 1 \to b \in F$. Thus, by Definition 4, $((1 \otimes c) \to (b \otimes c)) = c \to (b \otimes c) \in F$ and $((c \otimes 1) \to (c \otimes b)) = c \to (c \otimes b) \in F$.

(ii) $\Longleftrightarrow$ (iii): Assume (ii) and let $b \in F$. Hence, $c \to (b \otimes c) \in F$ which implies by applying item (ii) again, $d \to (d \otimes (c \to (b \otimes c))) \in F$. Since $\mathcal{E}$ is good, the converse follows by putting $d = 1$ and then $c = 1$.

(ii) $\Longrightarrow$ (i): Assume (ii), and let $a \to b \in F$. Hence, by the assumption, $(a \otimes c) \to ((a \to b) \otimes (a \otimes c)) \in F$. Thus, by associativity of $\otimes$, $(a \otimes c) \to (((a \to b) \otimes a) \otimes c) \in F$. Hence, by order properties of $\to$ and Lemma 2 (i), $(a \otimes c) \to (((a \to b) \otimes a) \otimes c) \leq (a \otimes c) \to (b \otimes c) \in F$. Similarly, we can prove that $(c \otimes a) \to (c \otimes b) \in F$. $\square$

We know that the underlying poset E, of an EQ-algebra $\mathcal{E}$ need not be a join-semilattice. Nevertheless, given $a, b \in E$, we shall write $a \vee b = 1$ meaning that the supremum of $\{a, b\}$ in E, exists and is equal to 1.

Proposition 10

Let $\mathcal{E}$ be a good EQ-algebra. Then, the following are equivalent, for all $a, b, c, d \in E$:

(i) $\mathcal{E}$ *is prelinear and satisfies the quasi-identity*

$$a \vee b = 1 \text{ implies } a \vee (d \to (d \otimes (c \to (b \otimes c)))) = 1 \tag{15}$$

(ii) $\mathcal{E}$ *satisfies the identity*

$$(a \to b) \vee (d \to (d \otimes (c \to ((b \to a) \otimes c)))) = 1 \tag{16}$$

(iii) $\mathcal{E}$ *satisfies*

$$(a \to b) \to u \leq [(d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u] \to u \tag{17}$$

(iv) $\mathcal{E}$ *satisfies*

$$(d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u \leq ((a \to b) \to u) \to u \tag{18}$$

PROOF: (i) $\Longleftrightarrow$ (ii): If $\mathcal{E}$ satisfies (16), then it satisfies $(a \to b) \vee (b \to a) = 1$; that is $\mathcal{E}$ is prelinear, which can be seen if we set $c = d = 1$. Let $a, b \in E$ such that $a \vee b = 1$. Hence, by Proposition 5, $a \to b = b$ and

$b \to a = a$, hence (16) gives $1 = (b \to a) \vee (d \to (d \otimes (c \to ((a \to b) \otimes c)))) = a \vee (d \to (d \otimes (c \to (b \otimes c))))$ for any $c, d \in E$. Thus, $\mathcal{E}$ satisfies (15). Conversely, If $\mathcal{E}$ is prelinear and satisfies (15), then it follows immediately that $\mathcal{E}$ satisfies (16).

(ii) $\iff$ (iii): We should notice that (16) is equivalent to (17) in the same way in which prelinearity is equivalent to (11), the proof runs as follows:

Assume (17) holds. Given $a, b, c, d \in E$, let δ be an upper bound in E of $\{(a \to b), (d \to (d \otimes (c \to ((b \to a) \otimes c))))\}$ (1 is one of those upper bounds). Then, by (17), we have $1 = ((a \to b) \to \delta) \leq ((d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to \delta) \to \delta = 1 \to \delta = \delta$; that is, δ must be 1. This shows that (16) holds.

Conversely, Assume (16) holds. It follows then $\mathcal{E}$ is prelinear and hence (by Theorem 4) it is good (and hence separated) ℓEQ-algebra. Denote $((a \to b) \to u) \to (((d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u) \to u)$ by θ. By Lemma 1 (EQ9), $((d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u) \to u \leq \theta$. Also, by EP of $\to$ (Lemma 2 (f)) and Lemma 1 (EQ9), $((a \to b) \to u) \to u \leq \theta$. Hence $(((a \to b) \to u) \to u) \vee (((d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u) \to u) \leq \theta$. Hence, by Lemma 7, $[((a \to b) \to u) \wedge ((d \to (d \otimes (c \to ((b \to a) \otimes c)))) \to u)] \to u \leq \theta$. Thus, by Lemma 2 (g) and the assumption (16), $1 = (1 \to u) \to u = u \to u \leq \theta$; that is (17) holds.

(iii) $\iff$ (iv): Direct by Lemma 2 (d). $\square$

A nonempty downward closed subset $I \subseteq E$ is called an *ideal* of a lattice-ordered EQ-algebra $\mathcal{E}$ if it is closed under finite joins. For each $a \in E$, set $F_a = \{b \in E : a \vee b = 1\}$. We shall use $(X]$ to denote the *downward closures* of a subset X of a partially ordered set.

We extend to separated lattice-ordered EQ-algebras the following result, proved by C. J. van Alten [2, Lemma 3.3] in the more special setting of residuated lattices. The proof is completely the same as Alten's proof. We shall supply the proof because of the importance of the statement and to make the paper self-contained:

Lemma 15
Let $\mathcal{E}$ be a separated lattice-ordered EQ-algebra. Then, for each $a \in \mathcal{E}$, F_a is a prefilter of $\mathcal{E}$. Moreover, if I is an ideal of $\mathcal{E}$, then $I' = \bigcup\{F_a : a \in I\}$ is a prefilter of $\mathcal{E}$.

PROOF: First, we will prove the second part of the lemma. It is obvious that $1 \in I'$. Now, let $c, c \to d \in I'$. Then for some $a, b \in I$, we have $a \vee c = 1 = b \vee (c \to d)$. Note that $a \vee b \in I$; we shall show that $(a \vee b) \vee d = 1$. Let $e \in E$ such that $a, b, d \leq e$. Then $c \to d \leq c \to e$. Also, $b \leq e \leq c \to e$ (by Lemma 1 (EQ9)), so $1 = b \vee (c \to d) \leq c \to e$, hence $c \leq e$ (by separation). Since $a \leq e$, we also have $1 = a \vee c \leq e$ so $e = 1$, hence $(a \vee b) \vee d = 1$. Thus, $d \in F_{(a \vee b)}$, as required. The first part of the lemma now follows from the observation that $(a]$ is an ideal and $F_a = (a]'$. $\square$

Lemma 16
Let F be a prefilter of a prelinear and separated ℓEQ-algebra $\mathcal{E}$. Then F is a minimal prime prefilter of $\mathcal{E}$ iff $F = \bigcup\{F_a : a \in E - F\}$.

PROOF: The analogous result in biresiduated lattices has been established by C. J. van Alten [2, Lemma 3.4]. The machinery employed in his proof consists of Lemma 15 and Proposition 8. So, his proof is valid in the present setting and applies verbatim here for prelinear and separated ℓEQ-algebras. □

Lemma 17
Let $\mathcal{E}$ be a good EQ-algebra. If $\mathcal{E}$ satisfies (17), or equivalently (18), then for each $a \in E$, F_a is a filter of $\mathcal{E}$ and, if I is an ideal of $\mathcal{E}$, then $\bigcup\{F_a : a \in I\}$ is a filter of $\mathcal{E}$. Thus, every minimal prime prefilter of $\mathcal{E}$ is a filter.

PROOF: Suppose that $\mathcal{E}$ satisfies (17). Hence, by Proposition 10, $\mathcal{E}$ is prelinear and satisfies (15). By Lemma 15, F_a is a prefilter. We will show that F_a satisfies condition (iii) of Proposition 9. Suppose that $b \in F_a$, i.e., $b \vee a = 1$, and let $c, d \in E$. By (15), we get $a \vee (d \to d \otimes (c \to b \otimes c)) = 1$, i.e., $(d \to d \otimes (c \to b \otimes c)) \in F_a$. Thus, F_a is a filter of $\mathcal{E}$. Moreover, if I is an ideal of $\mathcal{E}$, then clearly $\bigcup\{F_a : a \in I\}$ also satisfies condition (iii) of Proposition 9. For the last statement of the lemma, suppose that F is a minimal prime prefilter of $\mathcal{E}$ so, by Lemma 16, $F = \bigcup\{F_a : a \in E - F\}$. Since F is prime, $a \in E - F$ and $b \in E - F$ imply that $a \vee b \in E - F$ (by Proposition 8 (ii)). Therefore, $E - F$ is an ideal of $\mathcal{E}$ and the result follows from the first part of the lemma. □

We have settled all the auxiliary results, so we can prove the main goal as promised in the introduction:

Theorem 11
Let $\mathcal{E}$ be a good EQ-algebra. The following are equivalent:

(i) $\mathcal{E}$ is subdirectly embeddable into a product of linearly ordered good EQ-algebras; i.e., $\mathcal{E}$ is representable.

(ii) $\mathcal{E}$ satisfies (17), or equivalently (18).

(iii) $\mathcal{E}$ is prelinear and every minimal prime prefilter of $\mathcal{E}$ is a filter of $\mathcal{E}$.

PROOF: (i) $\implies$ (ii) It is obvious that if $\mathcal{E}$ is representable then it satisfies the identity (17), or equivalently (18) (since in linearly ordered good EQ-algebra one has either $x \to y = 1$ or $y \to x = 1$ for all x, y).

(ii) $\implies$ (iii): By Proposition 10, $\mathcal{E}$ is prelinear and hence it follows from Lemma 17 that every minimal prime prefilter of $\mathcal{E}$ is a filter of $\mathcal{E}$.

(iii) $\implies$ (i) Since $\mathcal{E}$ is prelinear, Lemma 14 (iv) holds for $\mathcal{E}$, hence the prime prefilters of $\mathcal{E}$ are precisely the meet-irreducible elements of $\mathbf{P}\mathcal{F}(\mathcal{E})$. Let X be the set of all minimal prime prefilters of $\mathcal{E}$. By Lemma 14 (iii), $\bigcap X = \{1\}$, hence, by Theorem 10, $\bigcap\{\approx_F : F \in X\}$ is the trivial congruence. Thus, by standard techniques of universal algebra (Cf. [6]), the natural homomorphism $h : \mathcal{E} \longrightarrow \prod_{F \in X} (\mathcal{E}/\approx_F)$ defined by $h(a) = \langle [a]_F \rangle_{F \in X}$ is a subdirect embedding of $\mathcal{E}$ into a direct product of $\{\mathcal{E}/\approx_F : F \in X\}$. Using Proposition 8 and Theorem 6, $\mathcal{E}/\approx_F$ is linearly ordered good EQ-algebra for each $F \in X$, which completes the proof. □

Although any of the (quasi-)identities or inequalities in Proposition 10 characterize the representable good EQ-algebra $\mathcal{E}$, we choose to use the identity (17), or equivalently (18), in order to avoid using $\vee$, since the underlying poset E of $\mathcal{E}$ need not be a join-semilattice.

Recall that Example 4 shows that the prelinearity alone does not characterize the class of representable good and commutative EQ-algebras. In the following we give a simple and an alternative characterization of the one provided in Theorem 11 for the class of representable good and commutative EQ-algebras.

Proposition 11
Let $\mathcal{E}$ be a good commutative EQ-algebra. Then, the following are equivalent, for all $a, b, c, d \in E$:

(i) $\mathcal{E}$ is prelinear and satisfies the quasi-identity (15).

(ii) $\mathcal{E}$ is prelinear and satisfies the quasi-identity

$$a \vee b = 1 \text{ implies } a \vee (c \to (b \otimes c)) = 1 \tag{19}$$

(iii) $\mathcal{E}$ satisfies the identity

$$(a \to b) \vee (c \to ((b \to a) \otimes c)) = 1 \tag{20}$$

(iv) $\mathcal{E}$ satisfies

$$(c \to ((b \to a) \otimes c)) \to u \leq ((a \to b) \to u) \to u \tag{21}$$

PROOF: (i) $\Longleftrightarrow$ (ii): Assume (ii) and let $a \vee b = 1$. Hence, by the commutativity of $\otimes$, we obtain (i). The converse is direct by putting $d = 1$ in (15) (since $\mathcal{E}$ is good and 1 is the identity element of $\otimes$).

(ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv): Similar to the proof of Proposition 10. □

In consequence, we get the following corollary:

Corollary 2
Let $\mathcal{E}$ be a good commutative EQ-algebra. The following are equivalent:

(i) $\mathcal{E}$ is representable.

(ii) $\mathcal{E}$ satisfies (21).

(iii) $\mathcal{E}$ is prelinear and every minimal prime prefilter of $\mathcal{E}$ is a filter of $\mathcal{E}$.

6. Conclusion

We continue in this paper the study of EQ-algebras, begun in [22], [23] and [8]. Following El-Zekey et al. [8], by EQ-algebras we mean those with multiplication not necessarily commutative while those with commutative multiplication, i.e. as in [23], are called here commutative EQ-algebras. We introduced prelinear EQ-algebras. We showed that every prelinear and good EQ-algebra is a good ℓEQ-algebra. Moreover, we showed that $\{\wedge, \vee, \rightarrow, 1\}$-reducts of prelinear and separated ℓEQ-algebra inherit several lattice-related properties from product of linearly ordered EQ-algebras.

We showed that prelinearity alone does not characterize the representable good (commutative) EQ-algebras. One of the main results of this paper is characterization of the representable good EQ-algebras. This is mainly based on the fact that $\{\rightarrow, 1\}$-reducts of good EQ-algebras are BCK-algebras and so, residuated lattices are " hidden" inside. We will focus more closely on this relation in subsequent papers. Our characterization is based on a detailed study of the prefilter lattice (see Section 4) and runs on lines of C. J. van Alten's [2] characterization of representable integral residuated lattices. We also supplied a number of potentially useful results, leading to this characterization.

Acknowledgement

I would like to thank to Vilem Novák for the repeated reading of the drafts of this paper and helping me to improve it a lot in many aspects.

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